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The Neoclassical Theory of Cooperatives: Mathematical Supplement

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This supplement presents mathematical expressions of the models of the farm supply cooperative described in Part I and the marketing cooperative described in Part II. Price and output solutions are derived for firms that maximize profit, cooperatives that maximize member returns, and cooperatives that handle whatever quantity of products members choose to purchase or deliver. Those solutions are then compared to the solutions for the maximization of economic welfare to determine the conditions under which profit-maximizing firms and cooperatives are efficient in an allocative sense.

Keywords: Cooperatives, farm supply cooperatives, marketing cooperatives, processing cooperatives, neoclassical theory, mathematical models, economic welfare

Introduction

Here mathematical models of a farm supply firm and a processing firm are presented to support the descriptive and graphical analyses included in Parts I and II. Price and output solutions are derived for the IOF (investor-owned firm) objective of maximizing profit and the cooperative objective of maximizing member returns. Solutions also are derived for cooperatives that handle whatever quantity of products members choose to purchase or deliver. Those solutions are then compared to the solutions for the maximization of economic welfare to determine the conditions under which profit-maximizing firms and cooperatives are efficient in an allocative sense. The material in this supplement should be appropriate for graduate students, advanced undergraduate students, and others with elementary skills in calculus.

A Model of a Farm Supply Firm

Assume that agricultural producers employ two inputs in the production of a single product according to the following production function:

$$q = q(x, y) \tag{1}$$

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where q is the quantity of the product and x and y represent the levels of the two inputs.[1] Producer profits can be represented as

$$\pi = p \cdot q(x, y) - r_x \cdot x - r_y \cdot y \tag{2}$$

where p is the price producers receive for the product and r_x and r_y are the prices they pay for inputs x and y.[2] Producers maximize profits according to the following first-order conditions:

$$\frac{\partial \pi}{\partial x} = p\frac{\partial q}{\partial x} - r_x = 0 \tag{3}$$

and

$$\frac{\partial \pi}{\partial y} = p\frac{\partial q}{\partial y} - r_y = 0 \tag{4}$$

where the terms $p(\partial q/\partial x)$ and $p(\partial q/\partial y)$ represent the marginal value products of x and y. To maximize profits, producers will employ each input at the level where its marginal value product is equal to its price.

Solving equations (3) and (4) simultaneously for x and y and summing over all producers yields the input demand functions:

$$x = x(r_x, r_y, p) \tag{5}$$

and

$$y = y(r_x, r_y, p). \tag{6}$$

The demand for each input is a function of the prices of both inputs and the output.[3]

Now consider a farm supply firm that specializes in the production of input x. Its profit can be defined as

$$\Pi = r_x(x) \cdot x - c(x) \tag{7}$$

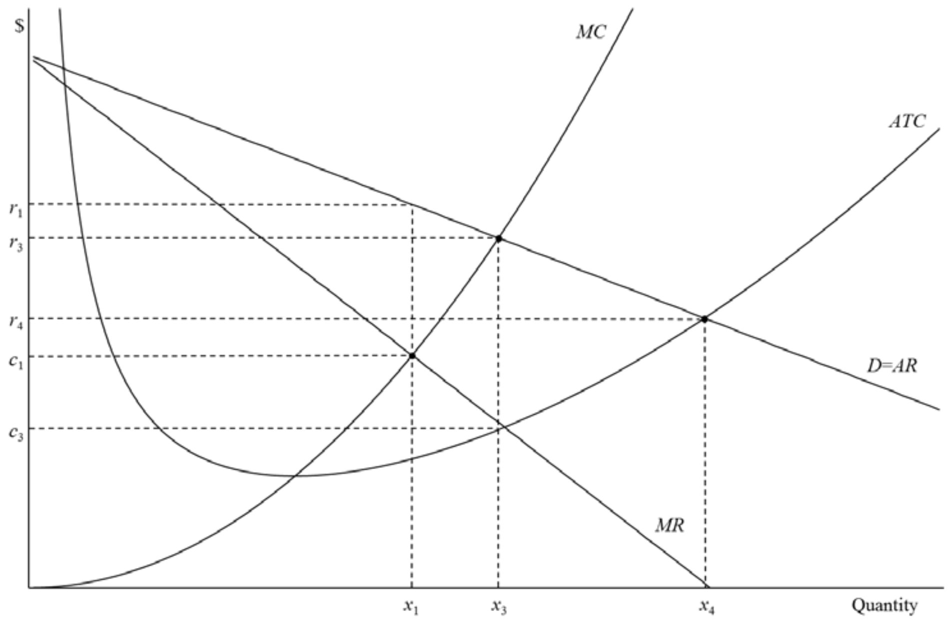

Figure 1. Price and output solutions for farm supply firms given a downward-sloping demand curve

where $r_x(x)$ is a convenient form for representing the inverse input demand function $r_x = r_x(x, r_y, p)$, which is determined by solving equation (5) for r_x in terms of *x*. The term $c(x)$ represents the total cost of producing *x*.

If the input supplier is a profit-maximizing firm, its first-order condition is

$$\frac{d\Pi}{dx} = \left[r_x(x) + x \cdot r_x'(x)\right] - c'(x) = 0, \tag{8}$$

which implies that the input supplier will maximize profit by producing *x* at the level where its marginal revenue from the sale of *x* is equal to the marginal cost of producing *x*,[4] represented by the quantity x_1 in figure 1.

Next consider a farm supply cooperative that maximizes member returns, including its own earnings, which are returned to members as patronage refunds. Assume all producers are members. Then the cooperative's objective function can be written

$$\begin{aligned}\Pi + \pi &= r_x(x) \cdot x - c(x) + p \cdot q(x,y) - r_x(x) \cdot x - r_y \cdot y \\ &= p \cdot q(x,y) - c(x) - r_y \cdot y\end{aligned} \tag{9}$$

where here π represents the sum of the profits of the individual producers in equation (2). The corresponding first-order condition is

$$\frac{d}{dx}(\Pi + \pi) = p\frac{\partial q}{\partial x} - c'(x) = 0 \tag{10}$$

where $p(\partial q/\partial x)$ once again represents the marginal value product of *x*. Thus the cooperative maximizes member returns by producing at the level where the marginal value product of *x* equals the marginal cost of producing *x*. From equation (3), we know that producers will operate such that the marginal value product of *x* is equal to the price paid for *x*. Thus

$$r_x(x) = c'(x) \tag{11}$$

is equivalent to the first-order in equation (10). The cooperative will produce at the level where the marginal cost of producing the farm input is equal to its market price, shown as x_3 in figure 1.

In the case of a cooperative that produces whatever quantity of *x* producers choose to purchase,[5] the receipt of patronage refunds provides producers an incentive to increase their purchases until the cooperative's average cost of producing *x* is equal to the price of *x* and the cooperative breaks even. Producers seek to maximize their profits:

$$\pi = p \cdot q(x,y) - (r_x - s) \cdot x - r_y \cdot y \tag{12}$$

where *s* represents the per-unit patronage refund and $r_x - s$ is the net price producers pay for the product. Their first-order conditions are

$$\frac{\partial \pi}{\partial x} = p\frac{\partial q}{\partial x} - (r_x - s) = 0 \tag{13}$$

and

$$\frac{\partial \pi}{\partial y} = p\frac{\partial q}{\partial y} - r_y = 0. \tag{14}$$

Solving equations (13) and (14) simultaneously for *x* and *y* and summing over all producers yields the input demand functions:

$$x = x(r_x - s, r_y, p) \tag{15}$$

and

$$y = y(r_x - s, r_y, p). \tag{16}$$

Solving equation (15) for $r_x - s$ in terms of *x*, we obtain the input demand function for *x* in its inverse form:

$$r_x - s = R_x(x, r_y, p). \tag{17}$$

The per-unit patronage refund *s* is equal to the cooperative's net earnings divided by the quantity of the farm input *x* it produces:

$$\begin{aligned} s &= \frac{r_x(x) \cdot x - c(x)}{x} \\ &= r_x(x) - c(x)/x. \end{aligned} \tag{18}$$

Substituting equation (18) for *s* in equation (17), we obtain the equilibrium condition for the cooperative:

$$r_x(x) - s = c(x)/x. \tag{19}$$

Equilibrium occurs where the net price of the farm input equals the average cost of producing it. For any particular net price, the values of r_x and *s* are not unique. Therefore, it is convenient to assume that the cooperative sets the cash price for the farm input equal to its average cost so that $r_x(x) = c(x)/x$ and $s = 0$. Substituting $s = 0$ into equation (19), the equilibrium condition can be expressed in a simpler form without loss of meaning:

$$r_x(x) = c(x)/x. \tag{20}$$

Equilibrium occurs where the price of the input *x* equals its average cost, represented by the quantity x_4 in figure 1.

A Model of a Processing Firm[6]

Assume producers produce a single raw product that is sold to a processor. Producers seek to maximize their profits:

$$\pi = r \cdot q - f(q) \tag{21}$$

where r is the raw product price paid producers by the processor, q is the quantity of raw product produced, and $f(q)$ is the total cost of producing the raw product. Profit maximization occurs where the marginal cost of producing the raw product equals the raw product price:

$$\frac{d\pi}{dq} = r - f'(q) = 0. \tag{22}$$

Solving equation (22) for r and summing over all producers yields the raw product inverse supply function $r = f'(q)$.

For convenience and without loss of generality, we can assume that a unit of processed product is equal to a unit of raw product. Then the processor's profit function can be written

$$\Pi = p(q) \cdot q - k(q) - r(q) \cdot q \tag{23}$$

where $p(q)$ is the processed product price and $k(q)$ represents total processing cost exclusive of the cost of the raw product. Here the raw product price is written as $r(q)$ to reflect the processor's monopsony power in the raw product market. Substituting the raw product inverse supply function for $r(q)$ in equation (23) and differentiating it with respect to quantity, the first-order condition for a profit-maximizing processor is

$$\frac{d\Pi}{dq} = \left[p(q) + q \cdot p'(q) \right] - k'(q) - \left[f'(q) + q \cdot f''(q) \right] = 0. \tag{24}$$

According to equation (24), a processor maximizes its profit by setting its marginal revenue in the processed product market equal to the sum of its marginal processing cost and the marginal factor cost of the raw product (*MFC*). The first two terms on the right, marginal revenue less the marginal processing cost, are equivalent the net marginal revenue product (*NMRP*). Thus the output of the

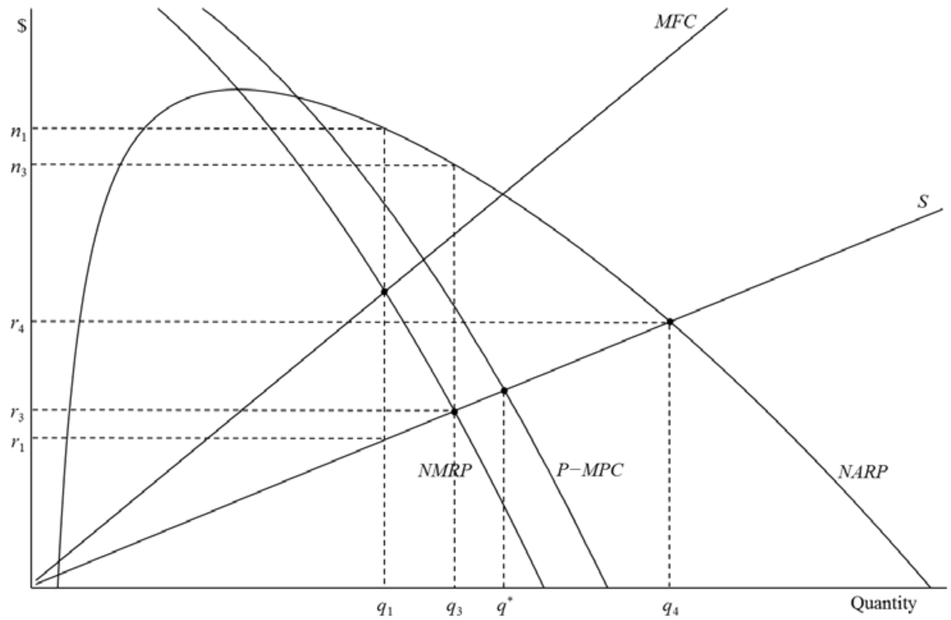

Figure 2. Price and output solutions for processing firms

profit-maximizing processor is q_1 in figure 2, determined by the intersection of the *NMRP* and *MFC* curves.

Now consider a cooperative processor that maximizes member returns, including its own earnings, which are returned to members as patronage refunds. Assume all producers are members. Then the cooperative's objective function can be written

$$\Pi + \pi = p(q) \cdot q - k(q) - f(q) \tag{25}$$

where here π represents the sum of the profits of the individual producers in equation (21). The corresponding first-order condition is

$$\frac{d}{dq}(\Pi + \pi) = \left[p(q) + q \cdot p'(q) \right] - k'(q) - f'(q) = 0. \tag{26}$$

The cooperative maximizes member returns by setting its marginal revenue in the processed product market equal to the sum of its marginal processing cost and the marginal cost of producing the raw product. The first two terms on the right are once again equivalent to *NMRP*. In addition, the last term is equivalent to the raw product supply curve according to equation (22). Thus the optimal level of output is q_3 in figure 2, determined by the intersection of the *NMRP* curve and the raw product supply curve *S*.

In the case of a cooperative that processes whatever quantity of raw product producers choose to deliver, the receipt of patronage refunds provides producers an incentive to expand output until the cooperative's net average revenue product (*NARP*) is equal to the raw product price and the cooperative breaks even, as in the Helmberger and Hoos (1962) model. Producers seek to maximize their profits:

$$\pi = (r+s)\cdot q - f(q) \tag{27}$$

where *s* represents the per-unit patronage refund. The first-order condition is

$$\frac{d\pi}{dq} = r + s - f'(q) = 0. \tag{28}$$

The per-unit patronage refund is equal to the cooperative's net earnings divided by the quantity of raw product processed:

$$\begin{aligned} s &= \frac{p(q)\cdot q - k(q) - r(q)\cdot q}{q} \\ &= p(q) - k(q)/q - r(q). \end{aligned} \tag{29}$$

Substituting equation (29) for *s* in equation (28), we obtain the equilibrium condition:

$$p(q) - k(q)/q - f'(q) = 0. \tag{30}$$

Equilibrium occurs where the processed product price less the average processing cost equals the marginal cost of producing the raw product. The first two terms are equivalent to *NARP*. Thus the output of a cooperative that processes whatever quantity members choose to deliver is determined by the intersection of the *NARP* and raw product supply curves, represented by the quantity q_4 in figure 2.

Maximization of Economic Welfare

Resources used in the production of a good are allocated efficiently if they are employed in such a manner that the economic welfare associated with its production and consumption is maximized. In the model of a farm supply firm, economic welfare consists of consumer surplus at the farm level:

$$CS_f = \int_0^{x^*} r_x(x)dx - r_x^* \cdot x^* \tag{31}$$

plus producer surplus at the supplier level:

$$PS_s = r_x^* \cdot x^* - \int_0^{x^*} c'(x)dx \tag{32}$$

where x^* and r_x^* are the quantity and price solutions for *x*. Summing equations (31) and (32), economic welfare can be written

$$W = \int_0^{x^*} \left[r_x(x) - c'(x) \right] dx. \tag{33}$$

Setting the first derivative to zero:

$$\frac{dW}{dx} = r_x(x) - c'(x) = 0. \tag{34}$$

Economic welfare is maximized at the level where the farm input price equals the marginal cost of producing the input, a well-known result, which is represented by the quantity x_3 in figure 1.

The first-order and equilibrium conditions for the various farm supply firms are compared to the welfare-maximizing condition in table 1. The first-order condition for a profit-maximizing firm differs from the welfare-maximizing condition in that it contains $r_x(x) + x \cdot r_x'(x)$, or marginal revenue, in place of $r_x(x)$, the farm input price. If the firm faces a downward-sloping demand curve, $x \cdot r_x'(x) < 0$. As a result, the marginal revenue curve will lie beneath the demand curve, and the firm will restrict its output to less than the welfare-maximizing level. Only if $r_x'(x) = 0$, i.e., the firm is a price taker, will the firm's production meet the criterion for allocative efficiency.

The first-order condition for a farm supply cooperative that maximizes member returns is identical to the welfare-maximizing condition. The cooperative

Table 1. Comparison of the output solutions for farm supply firms to the welfare-maximizing condition

Objective	Condition	Equation
Maximization of economic welfare	$r_x(x) = c'(x)$	(34)
Maximization of profit	$r_x(x) + x \cdot r_x'(x) = c'(x)$	(8)
Maximization of member returns (including patronage refunds)	$r_x(x) = c'(x)$	(11)
Production of quantity demanded by members	$r_x(x) = c(x)/x$	(20)

produces the optimal level of the farm input and uses resources efficiently. Examination of equation (20) reveals that this generally is not the case for a cooperative that produces whatever quantity of the farm input members choose to purchase. The equilibrium condition contains $c(x)/x$, the average cost of producing the input, in place of $c'(x)$, the marginal cost. If $c'(x) > c(x)/x$, the cooperative will overproduce x relative to the welfare-maximizing quantity because the marginal cost of producing x will exceed its value in producing the farm product q as reflected by its market price $r_x(x)$. The efficient level of x will be produced only if $c'(x) = c(x)/x$, as at the minimum of the *ATC* curve in figure 1 or under a cost structure characterized by constant marginal costs.

In the model of a processing firm, economic welfare consists of consumer surplus in the processed product market:

$$CS = \int_0^{q^*} p(q)dq - p^* \cdot q^* \qquad (35)$$

plus producer surplus at the processor level:

$$PS_p = p^* \cdot q^* - \int_0^{q^*} k'(q)dq - r^* \cdot q^* \qquad (36)$$

and producer surplus at the farm level:

Table 2. Comparison of the output solutions for processing firms to the welfare-maximizing condition

Objective	Condition	Equation
Maximization of economic welfare	$p(q) = k'(q) + f'(q)$	(39)
Maximization of profit	$p(q) + q \cdot p'(q) = k'(q) + f'(q) + q \cdot f''(q)$	(24)
Maximization of member returns (including patronage refunds)	$p(q) + q \cdot p'(q) = k'(q) + f'(q)$	(26)
Production of quantity supplied by members	$p(q) = k(q)/q + f'(q)$	(30)

$$PS_f = r^* \cdot q^* - \int_0^{q^*} f'(q)dq \tag{37}$$

where q^* is the quantity solution and p^* and r^* are respectively the processed and raw product price solutions. Summing equations (35), (36), and (37), economic welfare can be written

$$W = \int_0^{q^*} \left[p(q) - k'(q) - f'(q) \right] dq. \tag{38}$$

Setting the derivative to zero:

$$\frac{dW}{dq} = p(q) - k'(q) - f'(q) = 0. \tag{39}$$

Economic welfare is maximized at the level where the processed product price equals the sum of the marginal processing cost and the marginal cost of producing the raw product, represented by the quantity q^* in figure 2.

The first-order and equilibrium conditions for the various processing firms are compared to the corresponding welfare-maximizing condition in table 2. The first-order condition for a profit-maximizing firm differs from the welfare-maximizing condition in that it contains $p(q)+q\cdot p'(q)$, or marginal revenue in the processed product market, in place of $p(q)$, the processed product price, and it contains $f'(q)+q\cdot f''(q)$, the marginal factor cost of the raw product, in place of $f'(q)$, which is equivalent to the raw product price given equation (22). Thus a profit-maximizing firm will restrict output to a level less than the efficient level either if $p'(q)<0$, i.e., the firm faces a downward-sloping processed product demand curve, or if $f''(q)>0$, i.e., the firm faces an upward-sloping raw product supply curve. The firm will produce the efficient level of output only if $p'(q)=0$ and $f''(q)=0$, i.e., the firm is a price taker in both the raw and processed product markets.

The first-order condition for a cooperative that maximizes member returns differs from the welfare-maximizing condition only in that it contains $p(q)+q\cdot p'(q)$ in place of $p(q)$. Thus the cooperative will restrict output to less than the efficient level if $p'(q)<0$. If $p'(q)=0$, the two conditions are identical.

The equilibrium condition for a cooperative that processes whatever quantity of raw product members choose to deliver differs from the welfare-maximizing condition in that it contains $k(q)/q$, the average processing cost, in place of $k'(q)$, the marginal processing cost. If $k'(q)>k(q)/q$, the cooperative will overproduce q relative to the welfare-maximizing quantity because the sum of the marginal costs of producing and processing q will exceed its value to consumers as reflected by its price in the processed product market. The cooperative will produce the efficient level of output only if $k'(q)=k(q)/q$, as at the minimum of the average processing cost curve or under a cost structure characterized by constant marginal costs.

Notes

1. The purpose of assuming two inputs is to demonstrate that the demand for each input is a function of the price of the other input, as well as its own price and the price of the output. This model could easily be generalized to *n* inputs.

2. To keep the notation as simple as possible, we will not employ subscripts for individual agricultural producers.

3. For example, consider the production function

$$q = Ax^{\alpha}y^{\beta}$$

where $\alpha, \beta > 0$ and $\alpha + \beta < 1$. Substituting this function into equation (2) for q, we can derive the following first-order conditions:

$$\frac{\partial \pi}{\partial x} = p\alpha A x^{\alpha-1} y^{\beta} - r_x = 0$$

and

$$\frac{\partial \pi}{\partial y} = p\beta A x^{\alpha} y^{\beta-1} - r_y = 0.$$

Solving these conditions simultaneously for x and y, the input demand function for x is

$$x = \left[Ap\left(\frac{\alpha}{r_x}\right)^{1-\beta} \left(\frac{\beta}{r_y}\right)^{\beta} \right]^{\frac{1}{1-\alpha-\beta}}.$$

From this, it is clear that the demand for x is a function of both input prices and the output price.

4. Here and throughout, it is assumed that the second-order conditions for a maximum are satisfied. In this particular case, the first-order condition for a profit-maximizing input supplier can be rewritten

$$\frac{d\Pi}{dx} = MR - MC = 0$$

where MR and MC respectively represent the firm's marginal revenue and marginal cost. Consequently, the second-order condition for profit maximization can be written

$$\frac{d^2\Pi}{dx^2} = \frac{dMR}{dx} - \frac{dMC}{dx} < 0$$

or

$$\frac{dMR}{dx} < \frac{dMC}{dx}.$$

For a maximum, the slope of the marginal revenue curve must be less than the slope of the marginal cost curve, i.e., marginal cost must be increasing at a faster rate than marginal revenue.

5. This assumption is equivalent to assuming the cooperative maximizes the quantity of x it produces. Similarly, assuming a processing cooperative processes whatever quantity of raw product producers choose to deliver is equivalent to assuming it maximizes the quantity processed.

6. The models of a processing cooperative and the maximization of economic welfare are based on similar models presented in Royer (2001). As in Part II, the processor model can be applied to a cooperative that simply markets the raw product by considering the processing costs as representing the costs of transporting or marketing the product.

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